

# Paths and indices of maximal tail dependence

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**Abstract.** We demonstrate both analytically and numerically that the existing methods for measuring tail dependence in copulas may sometimes underestimate the extent of extreme co-movements of dependent risks and, therefore, may not always comply with the new paradigm of prudent risk management. This phenomenon holds in the context of both symmetric and asymmetric copulas with and without singularities. As a remedy, we introduce a notion of *paths of maximal (tail) dependence* and utilize it to propose several new indices of tail dependence. The suggested new indices are conservative, conform with the basic concepts of modern quantitative risk management, and are able to distinguish between distinct risky positions in situations when the existing indices fail to do so.

*Keywords and phrases:* multivariate distribution; copula; tail dependence; maximal dependence; fatal shock; multivariate Pareto; enterprise risk management.

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# 1 Introduction

Current regulatory frameworks require enhanced techniques for measuring and managing extremal risks of financial enterprises. In particular, this involves analyses of marginal risks and dependence structures between them (cf., e.g., Segal, 2011). The dependencies are frequently modelled using copulas, which have become a well established mathematical tool in actuarial and financial research and practice (cf., e.g., McNeil et al., 2005; Embrechts, 2009; Jaworski et al., 2010; Jaworski et al., 2013; and references therein), as well as in many other areas (cf., e.g., Balakrishnan and Lai, 2009).

Briefly, a bivariate function  $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is a copula if it is grounded, two-increasing, and have uniform marginals (cf., Nelsen, 2006). When it comes to the quantification of co-movements of extreme risks, the behaviour of  $C$  around the upper-right and lower-left vertices of the domain of definition  $[0, 1] \times [0, 1]$  give rise to indices of upper and lower tail dependence. While these indices are concerned with phenomena in opposing risk-tails, they can be formally unified by shifting the attention from the underlying copula to the corresponding survival copula. For example, given the index of lower-tail dependence  $\lambda_L := \lambda_L(C)$ , which is

$$\lambda_L = \lim_{u \downarrow 0} \frac{C(u, u)}{u}, \quad (1.1)$$

the corresponding index of upper-tail dependence  $\lambda_U := \lambda_U(C)$  is

$$\lambda_U = \lim_{u \downarrow 0} \frac{1 - 2(1 - u) + C(1 - u, 1 - u)}{u} = \lim_{u \downarrow 0} \frac{\widehat{C}(u, u)}{u},$$

where  $\widehat{C}$  is the survival copula of  $C$ ; hence,  $\lambda_U(C) = \lambda_L(\widehat{C})$ . Similar duality results hold for other indices of tail dependence, such as the index of weak lower-tail dependence  $\chi_L := \chi_L(C)$  given by

$$\chi_L = \lim_{u \downarrow 0} \frac{2 \log u}{\log C(u, u)} - 1, \quad (1.2)$$

and its upper variant  $\chi_U := \chi_U(C) = \chi_L(\widehat{C})$  (cf., Coles et al., 1999; Heffernan, 2000; Fischer and Klein, 2007; and references therein). In view of these notes, we restrict our following considerations to the behaviour of copulas near the lower-left vertex of their domain of definition.

Instead of comparing  $C(u, u)$  and  $u \in (0, 1)$  as in limits (1.1) and (1.2), we can more generally explore the asymptotic behaviour of  $C(u, u)$  in the form

$$C(u, u) = \ell(u)u^{\kappa_L} \quad \text{when } u \downarrow 0, \quad (1.3)$$

which defines the index  $\kappa_L := \kappa_L(C) \in [1, \infty)$  of lower-tail dependence, assuming of course that equation (1.3) holds for a slowly varying at 0 function  $\ell(u)$  (cf., Ledford and Tawn,

1996). We note that smaller values of  $\kappa_L$  correspond to stronger interdependences, thus motivating the search for the smallest possible analogues of  $\kappa_L$ , and this is our main goal in the present paper.

The rest of the paper is organized as follows. In Section 2 we lay down the main framework for our ideas, including the class of admissible dependence paths and its subclass of maximal dependence paths. In Section 3 we present a numerical example that demonstrates that the existing indices of tail dependence may underestimate the extent of extreme co-movements of dependent risks. In Section 4 we explore several families of copulas for which paths of maximal dependence can be derived in closed form, whereas somewhat more intricate examples will be discussed in Section 6. In Section 5, we further extend our general framework by introducing new tail orderings of distinct copulas. Section 7 concludes the paper. Tedious calculations are relegated to Appendix A.

## 2 Paths of maximal dependence and related indices

In what follows, we work with bivariate copulas  $C(u, v)$ ,  $0 \leq u, v \leq 1$ , only because they convey the main idea of the present paper in a simple and illuminating way. With the notation  $R(u, v) = [0, u] \times [0, v]$ , the copula-value  $C(u, v)$  is the probability that the bivariate random vector  $(U, V)$  with uniform marginals  $U$  and  $V$  falls into the rectangle  $R(u, v)$ , that is,

$$C(u, v) = \mathbf{P}[(U, V) \in R(u, v)]. \quad (2.1)$$

The class of all rectangles  $R(u, v)$  contains the subclass of all squares  $R(u, u)$ , and these are the ones that have traditionally been used for measuring the strength of tail dependence. Namely, the classical indices of lower-tail dependence are based on the behaviour of probability (2.1) when the rectangle  $R(u, v)$  shrinks along the diagonal

$$\{(u, v) \in [0, 1]^2 : u = v\} \quad (2.2)$$

when  $u \downarrow 0$ . (Throughout the paper, we work only with diagonal (2.2) and thus call it *the* diagonal.) In the context of the present paper, the diagonal is a path of (tail) dependence, but there are of course many other possible paths and we next describe them.

In order to make the problem well posed, some restrictions on the class of possible paths must be imposed. First, we observe that for the independence copula, it is natural to require that every path would reflect the same degree of tail dependence, that is, would have the same probability (2.1). For the functions  $\varphi, \psi : [0, 1] \rightarrow [0, 1]$ , this implies that every path  $(\varphi(u), \psi(u))_{0 \leq u \leq 1}$  that we consider has to be necessarily of the form  $(\varphi(u), u^2/\varphi(u))_{0 \leq u \leq 1}$ , that is, we must have  $\psi(u) = u^2/\varphi(u)$ . Certainly, both  $\varphi(u)$  and  $u^2/\varphi(u)$  must be in the interval  $[0, 1]$ , and thus  $\varphi(u)$  must be in the interval  $[u^2, 1]$ ,

which justifies the first property of the following definition. The second property of the definition will be related to the fact that we are interested in the behaviour of copulas near the lower-left vertex of their domain of definition.

**Definition 2.1.** We call a function  $\varphi : [0, 1] \rightarrow [0, 1]$  admissible if it satisfies the following two properties:

- (1)  $\varphi(u) \in [u^2, 1]$  for every  $u \in [0, 1]$ ; and
- (2) both  $\varphi(u)$  and  $u^2/\varphi(u)$  converge to 0 when  $u \downarrow 0$ .

We call the path  $(\varphi(u), u^2/\varphi(u))_{0 \leq u \leq 1}$  admissible whenever the function  $\varphi$  is admissible. Finally, we use  $\Phi$  to denote the set of all admissible functions  $\varphi$ .

Motivated by the idea of determining the strongest extreme co-movements of risks, among all admissible functions  $\varphi \in \Phi$ , we search for those that maximize the probability

$$\Pi_\varphi(u) = C(\varphi(u), u^2/\varphi(u)) \quad (2.3)$$

or, equivalently, the distance function

$$d_\varphi(C, C^\perp)(u) = C(\varphi(u), u^2/\varphi(u)) - C^\perp(\varphi(u), u^2/\varphi(u)), \quad (2.4)$$

where  $C^\perp$  is the independence copula, i.e.,  $C^\perp(u, v) = uv$  for all  $0 \leq u, v \leq 1$ . Obviously, the function  $\varphi_0(u) = u$  is admissible and yields the representation of the diagonal path that serves as a building block for the classical indices of lower-tail dependence. However, this path may not maximize probability (2.3), as we illustrate in following Example 2.1. In fact, in view of (2.4), the classical index  $\kappa_L$  may serve as neither a maximal nor minimal measure of tail dependence of the copula  $C$ .

**Example 2.1.** Consider the Marshall-Olkin copula

$$C_{a,b}(u, v) = \min(u^{1-a}v, uv^{1-b}) \quad \text{for } 0 \leq u, v \leq 1, \quad (2.5)$$

where  $a, b \in [0, 1]$  are parameters. We check that  $C_{a,b}(u, u) = \min(u^{2-a}, u^{2-b})$  and thus

$$\Pi_{\varphi_0}(u) = u^{\kappa_L} \quad \text{with} \quad \kappa_L = 2 - \min\{a, b\}.$$

Next we choose the admissible function  $\varphi_1(u) = u^{2b/(a+b)}$ . See Figure 2.1 for the corresponding path, which we later show to be a path of maximal dependence. We check that

$$\Pi_{\varphi_1}(u) = u^{\kappa_L^*} \quad \text{with} \quad \kappa_L^* = 2 - \frac{2ab}{a+b}. \quad (2.6)$$

Clearly,  $\kappa_L^* \leq \kappa_L$  with the equality holding only when  $a = b$ , that is, when the Marshall-Olkin copula is symmetric.

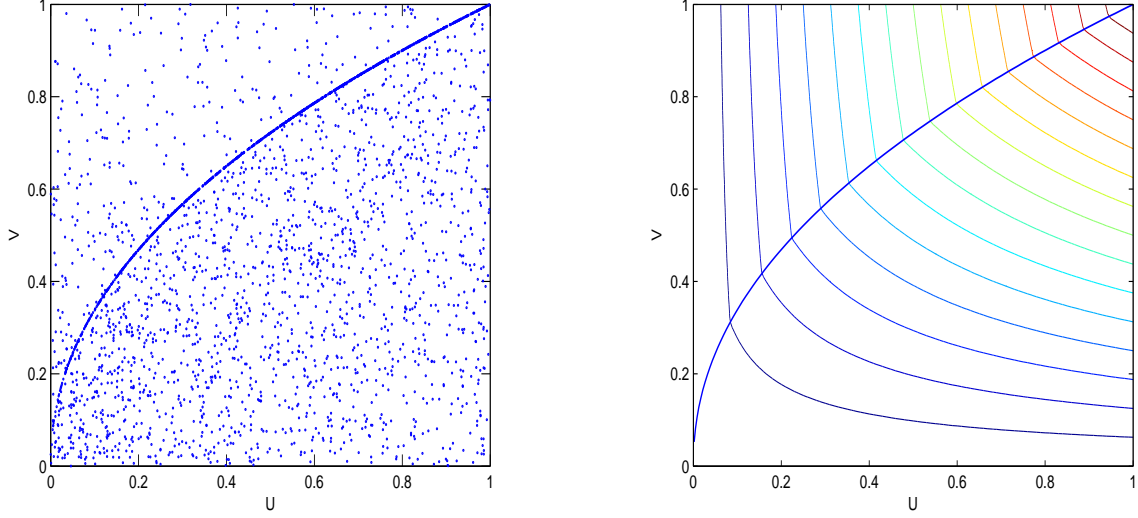


Figure 2.1: Simulated Marshall-Olkin copula (left) and its contours (right) when  $a = 0.3529$  and  $b = 0.75$ , with the path of maximal dependence  $(\varphi_1(u), u^2/\varphi_1(u))_{0 \leq u \leq 1}$  superimposed on the right-hand panel.

The example motivates the following definition of a subclass of admissible functions.

**Definition 2.2.** Given a copula  $C$ , an admissible function  $\varphi^* \in \Phi$  is called a function of maximal dependence, if

$$\Pi_{\varphi^*}(u) = \max_{\varphi \in \Phi} \Pi_{\varphi}(u) \quad \text{for all } u \in [0, 1].$$

We conveniently use the simpler notation  $\Pi^*(u)$  instead of  $\Pi_{\varphi^*}(u)$ , but may enhance it to  $\Pi^*(u \mid C)$  when a need arises to emphasize the dependence of  $\Pi^*(u)$  on  $C$ . Also, an admissible path  $(\varphi^*(u), u^2/\varphi^*(u))_{0 \leq u \leq 1}$  is referred to as a path of maximal dependence.

We next employ Definition 2.2 to introduce three conservative variants of the classical indices  $\lambda_L$ ,  $\chi_L$  and  $\kappa_L$ . Namely, assuming that the limits below exist, we set

$$\lambda_L^* := \lambda_L^*(C) = \lim_{u \downarrow 0} \frac{\Pi^*(u)}{u}, \quad (2.7)$$

$$\chi_L^* := \chi_L^*(C) = \lim_{u \downarrow 0} \frac{2 \log u}{\log \Pi^*(u)} - 1,$$

and let  $\kappa_L^* := \kappa_L^*(C)$  be such that

$$\Pi^*(u) = \ell^*(u) u^{\kappa_L^*} \quad \text{when } u \downarrow 0 \quad (2.8)$$

for a slowly varying at 0 function  $\ell^*(u)$ , assuming that such a function exists. In our illustrative examples below, we concentrate on calculating  $\kappa_L^*$  only.

We conclude this section by noting that this paper is not the first attempt to abandon the diagonal section of copulas when measuring tail dependence. For example, in their research of large claims reinsurance, Asimit and Jones (2008) rely on the asymptotic behaviour of copulas along non-diagonal paths. Asimit and Badescu (2010) rely on the extreme behaviour of copulas along a variety of paths when exploring a time dependent risk model with dependent inter-claim times and claim amounts. Joe et al. (2010) introduce the tail dependence function  $b(w_1, w_2; C) = \lim_{u \downarrow 0} C(uw_1, uw_2)/u$  for  $w_1 > 0$  and  $w_2 > 0$ . We refer to Asimit et al. (2011), Weng and Zhang (2012), and Li and Wu (2013) for further developments on the topic. Hua and Joe (2014) use the excess-of-loss economic pricing functional to propose and study a measure of tail dependence that does not rely on the diagonal dependence path. There are also several other related works but none of them – due to different research goals – aim at maximal-dependence paths and, in turn, at indices of maximal dependence.

### 3 Numerical illustration

Here we present a numerical example that questions the decisive role of the diagonal path in measuring tail dependence in copulas. We nevertheless stress at the outset that our discussion is not a criticism of the role of the diagonal when investigating copulas – it does play a pivotal role in the analysis of a variety of other aspects as elucidated by, e.g., Durante et al. (2014) and references therein.

Let  $X$  and  $Y$  be two random variables, which we assume to follow the same Pareto-II (also known as Lomax) distribution. Its decumulative distribution and probability density functions are given by, respectively,

$$\bar{F}(x) = \left( \frac{x - \mu}{\sigma} + 1 \right)^{-\alpha} \quad \text{and} \quad f(x) = \frac{\alpha}{\sigma} \left( \frac{x - \mu}{\sigma} + 1 \right)^{-(\alpha+1)} \quad (3.1)$$

for all  $x \geq \mu \in (-\infty, \infty)$ . Throughout this section, we set  $\mu = 0$  and  $\sigma = 1$  for simplicity, and set  $\alpha = 4$  to ensure the finiteness of all quantities that we consider.

Let the dependence structure between  $X$  and  $Y$  be given by the Marshall-Olkin copula; see equation (2.5). Hence, the joint cumulative distribution function of  $X$  and  $Y$  is equal to  $C_{a,b}(F(x), F(y))$ ,  $0 \leq x, y < \infty$  (for applications of this model to insurance, we refer to, e.g., Asimit et al., 2010). Following Embrechts et al. (2003), we set  $a = 0.3529$ . For several values of  $b \in [0, 1]$ , we calculate

- Kendall's  $\tau = \tau(C_{a,b})$  index of dependence
- classical index  $\kappa_L = \kappa_L(C_{a,b})$  of lower-tail dependence given by equation (1.3)

- newly suggested index  $\kappa_L^* = \kappa_L^*(C_{a,b})$  defined by equations (2.6) and (2.8)
- value-at-risk  $\text{VaR}_q[Z] = \inf\{x \in \mathbf{R} : F_Z(x) \geq q\}$  for  $Z = X + Y$
- conditional tail expectation  $\text{CTE}_q[Z] = \mathbf{E}[Z | Z > \text{VaR}_q[Z]]$
- modified tail variance

$$\text{MTVar}_q[Z] = \text{CTE}_q[Z] + \frac{1}{\text{CTE}_q[Z]} \text{Var}[Z | Z > \text{VaR}_q[Z]]$$

where  $q \in (0, 1)$ , which for all of the aforementioned weighted risk measures (cf., e.g., Furman and Zitikis, 2010; and references therein) is set to 0.990 and 0.995 in our numerical calculations. Results are summarized in Table 3.1.

Table 3.1: Indices of dependence and tail-based risk measures for the Marshall-Olkin copula with Pareto-II marginals.

Parameters		Indices of dependence			Risk measures		
$q$	$b$	$\tau$	$\kappa_L$	$\kappa_L^*$	$\text{VaR}_q[Z]$	$\text{CTE}_q[Z]$	$\text{MTVar}_q[Z]$
0.9900	0.7500	0.3158	1.6471	1.5200	3.4621	4.8599	5.5808
	0.5000	0.2609	1.6471	1.5862	3.4095	4.7606	5.4691
	0.3529	0.2143	1.6471	1.6471	3.3612	4.6926	5.3951
0.9950	0.7500	0.3158	1.6471	1.5200	4.2925	5.8976	6.7004
	0.5000	0.2609	1.6471	1.5862	4.2114	5.7782	6.5552
	0.3529	0.2143	1.6471	1.6471	4.1460	5.6801	6.4268

Noting that smaller values of  $\kappa_L^*$  mean stronger tail dependence, it is illuminating to observe from Table 3.1 that the smaller the values of  $\kappa_L^*$  are, the larger the values of VaR, CTE and MTVar are, whereas the classical index  $\kappa_L$  does not change.

## 4 Examples

We start with several families of copulas for which paths of maximal dependence are derivable in closed form.

### 4.1 Marshall-Olkin copula

Recall that the Marshall-Olkin copula is defined by formula (2.5). Next, for every  $u \in [0, 1]$ , the function  $x \mapsto C_{a,b}(x, u^2/x)$  defined on the interval  $[u^2, 1]$  is equal to  $u^{2(1-b)}x^b$

for all  $x \leq x_0 = u^{2b/(a+b)}$  and  $u^2/x^a$  for all  $x \geq x_0$ . Hence, the unique maximum of the function is achieved at the point  $x = x_0$ , and thus the function of maximal dependence of the Marshall-Olkin copula is unique and given by

$$\varphi^*(u) = u^{2b/(a+b)}. \quad (4.1)$$

Consequently, the maximal probability is

$$\Pi^*(u) = u^{2-2ab/(a+b)}, \quad (4.2)$$

and thus the lower-tail index of maximal dependence is

$$\kappa_L^* = 2 - \frac{2ab}{a+b}. \quad (4.3)$$

## 4.2 Mixture of Marshall-Olkin copulas

We see from formula (4.1) that the path of maximal dependence is diagonal if and only if  $a = b$ , and thus if and only if the Marshall-Olkin copula is symmetric. We next show that this fact cannot be generalized to arbitrary symmetric copulas, because we show next that there are symmetric copulas whose paths of maximal dependence are not diagonal.

To proceed, we use the mixture of two Marshall-Olkin copulas given by

$$C(u, v) = \frac{1}{2} \left( C_{a,b}(u, v) + C_{b,a}(u, v) \right) \quad \text{for } 0 \leq u, v \leq 1, \quad (4.4)$$

with some  $a \neq b$ . The copula  $C(u, v)$  is symmetric. Tedious calculations given in Appendix A show that there are two paths of maximal dependence (Figure 4.1) given by the functions

$$\varphi_1^*(u) = u^{2b/(a+b)} \quad \text{and} \quad \varphi_2^*(u) = u^{2a/(a+b)}. \quad (4.5)$$

None of the paths coincides with the diagonal because  $a \neq b$ , and for any of the two paths, the maximal probability is

$$\begin{aligned} \Pi^*(u) &= \frac{1}{2} u^{2-2ab/(a+b)} \left( 1 + u^{2(ab-\min\{a,b\})/(a+b)} \right) \\ &= \frac{1}{2} u^{2-2ab/(a+b)} (1 + o(1)) \quad \text{when } u \downarrow 0. \end{aligned} \quad (4.6)$$

Hence, the lower-tail index of maximal dependence is

$$\kappa_L^* = 2 - \frac{2ab}{a+b}, \quad (4.7)$$

which is the same as in equation (4.3) for the ordinary Marshall-Olkin copula.



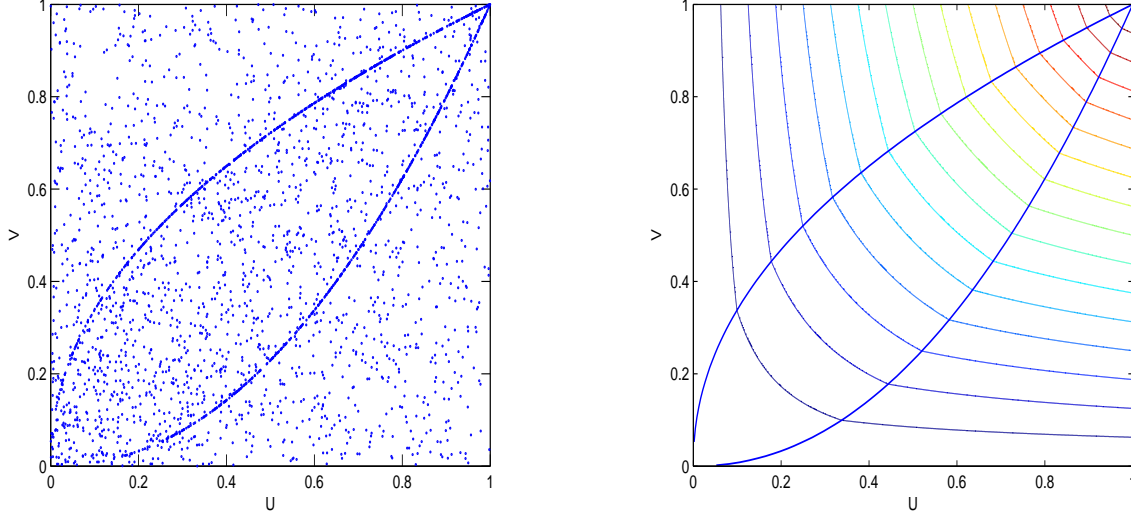


Figure 4.1: Simulated mixture of two Marshall-Olkin copulas (left) and its contours (right) when  $a = 0.3529$  and  $b = 0.75$ , with the two paths of maximal dependence  $(\varphi_k^*(u), u^2/\varphi_k^*(u))_{0 \leq u \leq 1}$ ,  $k = 1, 2$ , superimposed on the right-hand panel.

### 4.3 Farlie-Gumbel-Morgenstern copula

We saw in the previous subsection that the symmetry of a copula does not necessarily imply that its path of maximal dependence is diagonal. Of course, some symmetric copulas do have diagonal paths of maximal dependence. One of such examples is the already discussed Marshall-Olkin copula when  $a = b$ . Another example is the Farlie-Gumbel-Morgenstern (FGM) copula

$$C(u, v) = uv(1 + \alpha(1 - u)(1 - v)) \quad \text{for } 0 \leq u, v \leq 1,$$

with parameter  $\alpha \in [-1, 1]$ . Our choice to include the FGM copula among illustrative examples has been motivated by the fact that within the context of the present paper, it sometimes matters whether a copula is or is not positively quadrant dependent (PQD). Specifically for the FGM copula, when  $\alpha < 0$ , then the copula is negatively quadrant dependent (NQD); when  $\alpha > 0$ , then it is PQD; and when  $\alpha = 0$ , then it is the independence copula. We see the role of the sign of  $\alpha$  in our following considerations.

To find the path of maximal dependence for the FGM copula, for every  $u \in [0, 1]$ , we first search for those  $x \in [u^2, 1]$  that solve the equation  $(\partial/\partial x)C(x, u^2/x) = 0$ . Since

$$\begin{aligned} \frac{\partial C(x, u^2/x)}{\partial x} &= \frac{\partial}{\partial x} u^2 (1 + \alpha(1 - u^2/x)(1 - x)) \\ &= u^2 \alpha (-1 + u^2/x^2), \end{aligned}$$

we have  $(\partial/\partial x)C(x, u^2/x) = 0$  if and only if  $x = u$ . Therefore, when  $\alpha > 0$ , then the diagonal is the unique path of maximal dependence, that is,  $\varphi^*(u) = u$ . In this case,

the maximal probability is  $\Pi^*(u) = u^2(1 + \alpha(1 - u)^2)$ , and thus the lower-tail index of maximal dependence is

$$\kappa_L^* = 2.$$

As to the case  $\alpha \leq 0$ , we first note that when  $\alpha = 0$ , then every admissible path is a path of maximal dependence. When  $\alpha < 0$ , then  $C(x, u^2/x)$  reaches its maximal value at either  $u^2$  or 1 or both, but neither of the functions  $\varphi(u) = u^2$  and  $\varphi(u) = 1$  is admissible because they fail to satisfy property (2) of Definition 2.1.

## 5 Lower-tail dependence comparisons

We now look at the index  $\lambda_L^*$  (cf. equation (2.7)) from a different angle. To this end we first note that for the Fréchet upper bound copula  $C^\top(u, v) = \min\{u, v\}$ ,  $0 \leq u, v \leq 1$  the path of maximal dependence is diagonal. Consequently,  $\Pi^*(u | C^\top) = u$  and so, given any copula  $C$ , the index  $\lambda_L^* := \lambda_L^*(C)$  can be rewritten as

$$\lambda_L^*(C) = \lim_{u \downarrow 0} \frac{\Pi^*(u | C)}{\Pi^*(u | C^\top)}.$$

This suggests that, given two copulas  $C_1$  and  $C_2$ , we can compare their lower-tail dependencies using the index

$$\lambda_L^*(C_1, C_2) = \lim_{u \downarrow 0} \frac{\Pi^*(u | C_1)}{\Pi^*(u | C_2)}.$$

Obviously,  $\lambda_L^*(C) = \lambda_L^*(C, C^\top)$ .

**Definition 5.1.** *The copula  $C_1$  is said to be more (less) lower-tail maximally-dependent (LTMD) than the copula  $C_2$  if  $\lambda_L^*(C_1, C_2) > 1$  ( $< 1$ ).*

An analogous interpretation can be made about the index  $\chi_L^* := \chi_L^*(C)$ , but now using the independence copula  $C^\perp(u, v) = uv$ ,  $0 \leq u, v \leq 1$ , for which every admissible path is a path of maximal dependence. Hence,

$$\chi_L^*(C) = \lim_{u \downarrow 0} \frac{\log \Pi^*(u | C^\perp)}{\log \Pi^*(u | C)} - 1.$$

We generalize this index to the case of two arbitrary copulas  $C_1$  and  $C_2$  by introducing the index

$$\chi_L^*(C_1, C_2) = \lim_{u \downarrow 0} \frac{\log \Pi^*(u | C_2)}{\log \Pi^*(u | C_1)} - 1.$$

Obviously,  $\chi_L^*(C) = \chi_L^*(C, C^\perp)$ .

**Definition 5.2.** *The copula  $C_1$  is said to be more (less) weakly lower-tail maximally-dependent (WLTMD) than the copula  $C_2$  if  $\chi_L^*(C_1, C_2) > 0$  ( $< 0$ ).*

We can now compare all the specific copulas considered in this paper, as well as other ones. Without getting too specific at the moment, assume that  $C_1$  and  $C_2$  are such that, for some constants  $c^*(C_k)$  and  $\kappa_L^*(C_k)$ ,  $k = 1, 2$ , the asymptotic formulas  $\Pi^*(u | C_1) = c^*(C_1)u^{\kappa_L^*(C_1)}(1 + o(u))$  and  $\Pi^*(u | C_2) = c^*(C_2)u^{\kappa_L^*(C_2)}(1 + o(u))$  hold when  $u \downarrow 0$ . Two cases follow:

- When  $\kappa_L^*(C_1) = \kappa_L^*(C_2)$ , then

$$\lambda_L^*(C_1, C_2) = \frac{c^*(C_1)}{c^*(C_2)}.$$

To illustrate, we note that the classical Marshall-Olkin copula, which we denote here by  $C_{\text{MO}}$ , and mixture (4.4) of two Marshall-Olkin copulas, which we denote here by  $C_{\text{MIXMO}}$ , have the same  $\kappa_L^*$ -indices. Equations (4.2) and (4.6) imply that  $\lambda_L^*(C_{\text{MO}}, C_{\text{MIXMO}}) = 2$ , which we interpret as saying that  $C_{\text{MO}}$  is more LTMD than  $C_{\text{MIXMO}}$  whenever  $a \neq b$ . When  $a = b$ , then we of course have  $\lambda_L^*(C_{\text{MO}}, C_{\text{MIXMO}}) = 1$  because  $C_{\text{MIXMO}} = C_{\text{MO}}$  in this case.

- When  $\kappa_L^*(C_1) \neq \kappa_L^*(C_2)$ , then only the index  $\chi_L^*(C_1, C_2)$  is of interest, which can be expressed by the formula

$$\chi_L^*(C_1, C_2) = \frac{\kappa_L^*(C_2)}{\kappa_L^*(C_1)} - 1.$$

In particular, we have  $\chi_L^*(C) = \chi_L^*(C, C^\perp) = 2/\kappa_L^*(C) - 1$ .

## 6 Further examples

For all the copulas hitherto, we have successfully derived closed-form expressions for the paths of maximal dependence. This is not always the case, and we demonstrate the fact in the following examples. Importantly, we still be able to derive closed-form expressions for the index  $\kappa_L^*$ .

### 6.1 Generalized Clayton copula

The generalized Clayton copula is given by the formula

$$C(u, v) = u^{\gamma_1/\tilde{\gamma}_1}(u^{-1/\tilde{\gamma}_1} + v^{-1/\gamma_0} - 1)^{-\gamma_0} \quad \text{for } 0 \leq u, v \leq 1,$$

where  $\gamma_0 > 0$  and  $\gamma_1 \geq 0$  are parameters, and  $\tilde{\gamma}_1 = \gamma_0 + \gamma_1$ . Tedious computations, which we have relegated to Appendix A, show that the function of maximal dependence  $\varphi^*$  (Figure 6.1) is unique and satisfies the equation

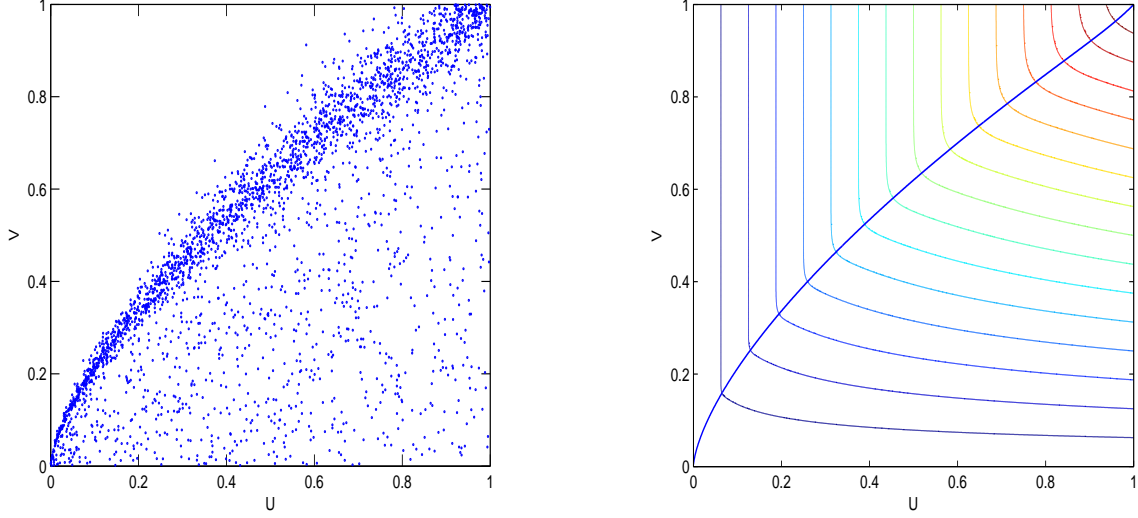


Figure 6.1: Simulated generalized Clayton copula (left) and its contours (right) when  $\gamma_0 = 0.04$  and  $\gamma_1 = 0.02$ , with the path of maximal dependence  $(\varphi^*(u), u^2/\varphi^*(u))_{0 \leq u \leq 1}$  superimposed on the right-hand panel.

$$\zeta(\varphi^*(u)) = 0 \quad (6.1)$$

for every  $0 \leq u \leq 1$ , where

$$\zeta(x) = x^{-1/\gamma_0} (x^{-1/\tilde{\gamma}_1} - (\gamma_1/\tilde{\gamma}_1)) - (1 - (\gamma_1/\tilde{\gamma}_1)) u^{-2/\gamma_0}$$

for all  $x \in [u^2, 1]$ . Further tedious calculations (Appendix A) show that the lower-tail index of maximal dependence is

$$\kappa_L^* = 1 + \frac{\gamma_1}{\gamma_1 + 2\gamma_0}. \quad (6.2)$$

## 6.2 Archimedean copula

Another example illustrating the difficulty of deriving closed-form expressions for paths of maximal dependence is provided by the Archimedean copula

$$C(u, v) = \psi^{-1}(\psi(u) + \psi(v)) \quad \text{for } 0 \leq u, v \leq 1, \quad (6.3)$$

where  $\psi : [0, 1] \rightarrow [0, \infty]$  is such that  $\psi(1) = 0$ ,  $\psi'(u) < 0$ , and  $\psi''(u) > 0$  for all  $0 < u < 1$ . Assume that the Archimedean generator  $\psi$  is strict, that is,  $\psi(u) \rightarrow \infty$  when  $u \downarrow 0$ , as only in this case can we have a path of maximal dependence. For every  $u \in [0, 1]$ , to determine those  $x \in [u^2, 1]$  for which the equation  $(\partial/\partial x)C(x, u^2/x) = 0$  holds, we notice that when the function  $x\psi'(x)$  is increasing on  $[u^2, 1]$ , which we assume from now on, then

$$x\psi'(x) - (u^2/x)\psi'(u^2/x) = \begin{cases} < 0 & \text{when } x < u^2/x, \\ > 0 & \text{when } x > u^2/x. \end{cases}$$

Thus,  $\psi(x) + \psi(u^2/x) \geq 2\psi(u)$  for all  $x \in [u^2, 1]$  or, equivalently,

$$\psi^{-1}(\psi(x) + \psi(u^2/x)) \leq \psi^{-1}(2\psi(u)).$$

Therefore, when the function  $x\psi'(x)$  is increasing on  $[u^2, 1]$ , then the diagonal is a path of maximal dependence. In this case, the Archimedean copula is PQD. We observe that all PQD Archimedean copulas documented by Nelsen (2006) have increasing functions  $x\psi'(x)$ .

## 7 Conclusion

We have demonstrated that the classical indices of tail dependence, which are based on the behaviour of copulas along the diagonal path, do not generally capture the maximal degree of tail dependence. For this reason, we have proposed conservative indices of tail dependence that hinge on the new notion of *paths of maximal dependence* that we have introduced herein. We have used a number of specific copulas, as well as a numerical example, to elucidate relevant main ideas both analytically and numerically. Our approach to assessing the tail dependence in copulas conforms with the new paradigm of prudence in modern quantitative risk management.

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## A Appendix: proofs

Throughout the proofs, though perhaps not mentioned explicitly,  $u$  and  $v$  are always in the interval  $[0, 1]$ , and  $x$  is always in the interval  $[u^2, 1]$ .

*Proof of equations (4.5) and (4.7).* Without loss of generality, let  $a < b$ . Then  $x_{a,b} = u^{2b/(a+b)}$  is smaller than  $x_{b,a} = u^{2a/(a+b)}$ , and so the two functions  $x \mapsto C_{a,b}(x, u^2/x)$  and  $x \mapsto C_{b,a}(x, u^2/x)$  are increasing on the interval  $(u^2, x_{a,b})$  and decreasing on  $(x_{b,a}, 1)$ . Hence, the maximum of the function  $x \mapsto C(x, u^2/x)$  can only be achieved on the interval  $[x_{a,b}, x_{b,a}]$ . In this interval, we have the formula

$$C(x, u^2/x) = \frac{1}{2} \left( u^{2(1-a)} x^a + u^2/x^a \right).$$

We split the interval  $[x_{a,b}, x_{b,a}]$  into two subintervals:  $[x_{a,b}, u]$  where the function  $x \mapsto C(x, u^2/x)$  is decreasing, and  $[u, x_{b,a}]$  where the function is increasing. From this we conclude that the function  $x \mapsto C(x, u^2/x)$  can achieve its maximum only at  $x_{a,b}$  and/or  $x_{b,a}$ , and it attains the same value at both end-points. Consequently, the two end-points are maxima and thus define two functions of maximal dependence, which are  $\varphi_1^*$  and  $\varphi_2^*$  given by formulas (4.5). For any of these two functions,  $\varphi_k^*$ , we have

$$C(\varphi_k^*(u), u^2/\varphi_k^*(u)) = \frac{1}{2} \left( u^{2-2(a \wedge b)^2/(a+b)} + u^{2-2ab/(a+b)} \right).$$

With  $\kappa = 2 - 2ab/(a+b)$  we check that

$$\frac{\Pi^*(u)}{u^\kappa} = \frac{C(\varphi_k^*(u), u^2/\varphi_k^*(u))}{u^\kappa} \rightarrow \frac{1}{2}$$

when  $u \downarrow 0$ . Consequently, the lower-tail index of maximal dependence  $\kappa_L^*$  is equal to  $\kappa$ . This concludes the proof of equations (4.5) and (4.7).  $\square$

*Proof of equations (6.1) and (6.2).* We start with the equation

$$C(x, u^2/x) = x^{\gamma_1/\tilde{\gamma}_1} (x^{-1/\tilde{\gamma}_1} + u^{-2/\gamma_0} x^{1/\gamma_0} - 1)^{-\gamma_0}. \quad (\text{A.1})$$

Finding the maximum of  $C(x, u^2/x)$  with respect to  $x \in [u^2, 1]$  is the same as finding the maximum of its logarithm  $\log C(x, u^2/x)$ , which is easier in this particular case. We have

$$\frac{\partial}{\partial x} \log(C(x, u^2/x)) = \frac{x^{-1/\tilde{\gamma}_1} + ((\gamma_1/\tilde{\gamma}_1) - 1)(u^2/x)^{-1/\gamma_0} - (\gamma_1/\tilde{\gamma}_1)}{x(x^{-1/\tilde{\gamma}_1} + (u^2/x)^{-1/\gamma_0} - 1)}.$$

The denominator  $x(x^{-1/\tilde{\gamma}_1} + (u^2/x)^{-1/\gamma_0} - 1)$  is positive for all  $x \in [u^2, 1]$ , and thus we need to find those  $x \in [u^2, 1]$  that make the numerator equal to 0. This is equivalent to solving the equation  $\zeta(x) = 0$ . The solution to this equation is unique because

$$\zeta(u^2) = u^{-2/\gamma_0} (u^{-2/\tilde{\gamma}_1} - 1) > 0,$$

$$\zeta(1) = (1 - (\gamma_1/\tilde{\gamma}_1)) - (1 - (\gamma_1/\tilde{\gamma}_1)) u^{-2/\tilde{\gamma}_1} < 0,$$

and, for all  $x \in [u^2, 1]$ ,

$$\zeta'(x) = -\frac{2}{\tilde{\gamma}_1} x^{-1/\tilde{\gamma}_1-1/\gamma_0-1} - \frac{\gamma_1}{\tilde{\gamma}_1\gamma_0} (x^{-1/\tilde{\gamma}_1-1/\gamma_0-1} - x^{-1/\gamma_0-1}) < 0.$$

We cannot derive a closed form solution to the equation  $\zeta(x) = 0$ , but we already know that the solution  $x = \varphi^*(u) \in [u^2, 1]$  exists and is unique. Furthermore, the solution satisfies the equation

$$x^{-1/\tilde{\gamma}_1-1/\gamma_0} (1 - (\gamma_1/\tilde{\gamma}_1)x^{1/\tilde{\gamma}_1}) = (1 - (\gamma_1/\tilde{\gamma}_1)) u^{-2/\gamma_0}. \quad (\text{A.2})$$

We also have that  $1 - (\gamma_1/\tilde{\gamma}_1)x^{1/\tilde{\gamma}_1} \in [1 - (\gamma_1/\tilde{\gamma}_1), 1]$ . From these facts we conclude that  $x = \varphi^*(u) \downarrow 0$  when  $u \downarrow 0$ .

Due to the lack of closed-form expression for the function of maximal dependence, we cannot obtain a closed-form expression for the maximal probability  $\Pi^*(u)$  either. Nevertheless, we can obtain a closed-form expression for  $\kappa_L^*$ . Starting with equation (A.2), we arrive at the following one:

$$x = \left( u^{2/\gamma_0} (1 - (\gamma_1/\tilde{\gamma}_1)x^{1/\tilde{\gamma}_1}) / (1 - (\gamma_1/\tilde{\gamma}_1)) \right)^{\tilde{\gamma}_1\gamma_0/(\tilde{\gamma}_1+\gamma_0)}. \quad (\text{A.3})$$

Denote  $r(x) = (1 - (\gamma_1/\tilde{\gamma}_1)x^{1/\tilde{\gamma}_1}) / (1 - (\gamma_1/\tilde{\gamma}_1))$ . Replacing all the  $x$ 's on the right-hand side of equation (A.1) by the right-hand side of equation (A.3), we obtain

$$\begin{aligned} C(x, u^2/x) &= \{u^{2/\gamma_0} r(x)\}^{\tilde{\gamma}_1\gamma_0/(\tilde{\gamma}_1+\gamma_0)} \\ &= u^{\frac{2\gamma_1}{\tilde{\gamma}_1+\gamma_0}} r(x)^{\frac{\gamma_0\gamma_1}{\tilde{\gamma}_1+\gamma_0}} \left( r(x)^{-\frac{\gamma_0}{\tilde{\gamma}_1+\gamma_0}} u^{-\frac{2}{\tilde{\gamma}_1+\gamma_0}} + r(x)^{\frac{\tilde{\gamma}_1/\gamma_0}{\tilde{\gamma}_1+\gamma_0}} u^{-\frac{2}{\tilde{\gamma}_1+\gamma_0}} - 1 \right)^{-\gamma_0}. \end{aligned}$$

Consequently, with  $\kappa = 1 + \gamma_1/(\tilde{\gamma}_1 + \gamma_0)$  we have

$$\frac{C(x, u^2/x)}{u^\kappa} = r(x)^{\frac{\gamma_0\gamma_1}{\tilde{\gamma}_1+\gamma_0}} \left( r(x)^{-\frac{\gamma_0}{\tilde{\gamma}_1+\gamma_0}} + r(x)^{\frac{\tilde{\gamma}_1/\gamma_0}{\tilde{\gamma}_1+\gamma_0}} - u^{\frac{2}{\tilde{\gamma}_1+\gamma_0}} \right)^{-\gamma_0} \rightarrow c \in (0, \infty)$$

when  $u \downarrow 0$ . This proves that the lower-tail index of maximal dependence  $\kappa_L^*$  is equal to  $\kappa$ . The proof of equations (6.1) and (6.2) is finished.  $\square$